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Some Remarks Concerning Stability for Nonstationary Quantum Systems

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The problem of characterizing stability and instability for general nonstationary quantum systems is investigated. Some characterizations are reported and some elementary properties of a topological characterization are established. Then, it is proven, by considering a simple example, that there are nonperiodic driven systems whose orbits are neither precompact nor leave on average any compact set. Autocorrelation measures are computed and the possible roles of the generalizes quasienergy operator and energy growth are briefly discussed.

KEY WORDS: Quantum stability; driven systems; precompact orbits; energy growth.

1. INTRODUCTION

In this paper we study the characterization of stable and unstable states for general nonstationary quantum systems. Little is known about the time evolution of quantum systems with time-dependent Hamiltonians, although it is a problem of significant interest.[1-24] The quantum time evolution is given by a strongly, continuous family of unitary operators (propagator) $U(t, r)$ acting on the separable Hilbert space $\mathscr{H}$ of quantum states, such that

$$U(t, r)\, U(r, s) = U(t, s)$$
$$U(t, t) = I \qquad \text{(identity operator)}$$

for all t, r, s. If, at $t = 0$, $\psi = \psi_0 \in \mathscr{H}$, its time evolution is given by the (weak) solution $U(t, 0)\psi_0$ of the Schrödinger equation. If the Hamiltonian is time-periodic with period T, then $U(t + T, r + T) = U(t, r)$ and we have the Floquet operator $U_F \equiv U(T, 0)$ at our disposal.[1,2,19] Here, we shall

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focus our attention on the long-time behavior of driven systems, i.e., the so-called *stability* problem.

The condition of quantum stability may be formulated in different ways, and aspects of this concept have been related to the RAGE theorem,[1] energy growth as function of time,[1,2,9,10,19,20] atomic ionization,[2,19] trajectories leaving any finite-dimensional subspace of the Hilbert space,[1,9] etc. These aspects are not unrelated, and the understanding of those relations is a problem of major interest.

Let U denote U_F in the periodic case and $U(t, 0) = e^{-iHt}$ in the autonomous case, where H is the Hamiltonian operator. If the underlying physical model is time-independent or time-periodic, the quantum stability problem has been reduced to the spectral analysis of U.[1-10,19,24] More precisely, the quantum stable subspace and unstable subspace have been assigned, respectively, to the point spectral subspace $\mathscr{H}_{\text{po}}(U)$ and the continuous subspace $\mathscr{H}_{\text{cont}}(U)$.

In the case of nonperiodic time dependence we do not have a clear spectral characterization of stability, and we are faced with the problem of giving a satisfactory definition. At least four different approaches to this problem have been proposed. The first one is to consider the autocorrelation function for nontrivial $\psi \in \mathscr{H}$,

$$C_\psi(t) \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \langle U(s, 0)\psi \mid U(t+s, 0)\psi \rangle \, ds$$

If this limit exists, one can apply Bochner's theorem[3,6,25] and conclude that there is a positive measure σ_ψ, the so-called autocorrelation measure, such that

$$C_\psi(m) = \int_0^{2\pi} e^{-i\lambda m} \, d\sigma_\psi(\lambda) \tag{1}$$

The autocorrelation measures are generalizations of spectral measures.[3] In this approach one says that ψ belongs to the stable subspace if σ_ψ is pure point, and to the unstable subspace if σ_ψ is pure continuous.

The second approach is to study the quasienergy operator,[8,9,19,21] i.e., a self-adjoint operator formally given by

$$K = -i \frac{\partial}{\partial t} + H(t) \tag{2}$$

acting in some enlarged Hilbert space. Here the stability problem can be summarized by asking if K has pure point spectrum. The quasienergy operator K was previously defined for periodic Hamiltonians,[8,21,23] and then generalized to Hamiltonians $H = H_0 + V(t)$ with $V(t) = V(\theta(t))$, where $\theta(t)$ is the trajectory of an invertible classical dynamical system having an invariant ergodic measure;[9] in the latter case the results are usually stated for typical sets with respect to that measure.[9,10] It has been found that when the autocorrelation of the potential $V(\theta(t))$ decays rapidly, the average energy grows linearly, and when the autocorrelations do not decay fast enough, the long-time behavior may depend on details of the particular model, e.g., on differentiability properties of the potential.[10,22] In the case of quasiperiodic time evolution (finite number of fundamental frequencies) there is also a natural generalization of the Floquet operator[9] and its spectrum is in one-to-one correspondence with the spectrum of the generalized quasienergy operator. If $V(t)$ is periodic, the spectral properties of K, or the associated Floquet operator, have been successfully used to characterize quantum stability and instability.[2,9,10,19,21-23] For related results in the case of quasiperiodic driven systems we refer to the work of Jauslin and Lebowitz.[9]

The third approach is motivated by many models whose Hamiltonians can be put in the form $H = H_0 + V(t)$, where H_0 is an unbounded discrete Hamiltonian.[1-7,10,19-23] Specific cases of interest include the harmonic and anharmonic oscillators, the rotor, and a particle in a finite box. The instability is associated to an unbounded absorption of energy of the unperturbed system from the perturbation $V(t)$, so that the expectation value of H_0, $\langle U(t, 0)\varphi \mid H_0 U(t, 0)\varphi\rangle$, becomes an unbounded function of time t. We then suggest analyzing the behavior of an "abstract energy" we shall represent by an unbounded positive self-adjoint operator A: dom $A \subset \mathscr{H} \to \mathscr{H}$, with pure point spectrum, $A\varphi_n = \lambda_n \varphi_n$, $0 \leqslant \lambda_n \leqslant \lambda_{n+1}$. For simplicity we shall avoid some involved domain questions by assuming that if $\varphi \in \operatorname{dom} A$, then $U(t, 0)\varphi \in \operatorname{dom} A$ for all $t \geqslant 0$, so that $E_\varphi^A(t) \equiv \langle U(t, 0)\varphi \mid AU(t, 0)\varphi\rangle$ is finite for any fixed $t \geqslant 0$. It is conceivable then that stability might be related to bounded functions $E_\varphi^A(t)$, and that stability results do not depend on the particular A one picks up (see below).

The last approach we comment upon is based on results by Enss and Veselic.[1] They addressed the question of possible generalizations of the RAGE theorem to the general nonperiodic case and gave a topological characterization of $\mathscr{H}_{\rm po}(U)$. More precisely, they proved that for time-periodic Hamiltonians (or time-independent models) $\mathscr{H}_{\rm po}(U)$ coincides with the space $\mathscr{H}^{\rm p}(U)$ of vectors with precompact (totally bounded) trajectories, and that $\mathscr{H}_{\rm cont}(U) = \mathscr{H}^{\rm f}(U)$, where

$$\mathscr{H}^{\rm f}(U) = \left\{ \psi \in \mathscr{H} \;\middle|\; \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \|\mathbf{K} U(t, 0)\psi\|^2 \, dt = 0 \text{ for any compact operator } \mathbf{K} \right\} \tag{3}$$

Important compact operators are the projectors onto finite subspaces of $\mathscr{H}$, so that the elements of $\mathscr{H}^{\mathrm{f}}$ are interpreted as the vectors that under time evolution leave—on average—any finite-dimensional subspace of $\mathscr{H}$. A significant class of compact operators here is given by spectral projectors onto finite-dimensional eigenspaces of the energy A (as defined above), and since (3) does not depend on the specific compact projector $\mathbf{K}$, we have a hint that the boundedness of $E_\varphi^A(t)$ should not depend on the particular energy A, although the situation is not clear yet.[9] It is then natural to take $\mathscr{H}^{\mathrm{p}}(U)$ and $\mathscr{H}^{\mathrm{f}}(U)$ as possible generalizations of the stable and unstable subspaces, respectively, even in the nonperiodic case.

The point of this paper is to initiate a discussion about the use of these approaches in the case of nonperiodic time dependence. We collect some subsets of $\mathscr{H}$ of interest in the following definition; recall that it is implicitly assumed that the operator A is positive, unbounded, and discrete, such that dom A is invariant under time evolution,

Definition 1. Let $U(t, r)$ be a propagator acting on the Hilbert space $\mathscr{H}$.

(i) $\mathscr{H}^{\mathrm{p}}(U) \equiv \{\psi \in \mathscr{H} \mid \{U(t,0)\psi \mid t \geqslant 0\}$ is precompact in $\mathscr{H}\}$.

(ii) $\mathscr{H}^{\mathrm{f}}(U) \equiv \{\varphi \in \mathscr{H} \mid \varphi$ satisfies the right-hand side of (3)$\}$.

(iii) $\mathscr{S}^{\mathrm{bd}}(A) \equiv \{\varphi \in \mathrm{dom}\, A \mid$ the function $t \to E_\varphi^A(t)$ is bounded$\}$.

(iv) $\mathscr{S}^{\mathrm{un}}(A) \equiv \mathrm{dom}\, A \cap \mathscr{S}^{\mathrm{bd}}(A)^{\perp}$.

We have simplified the definitions of $\mathscr{S}^{\mathrm{bd}}(A)$ and $\mathscr{S}^{\mathrm{un}}(A)$ by assuming that dom A is invariant under time evolution. In fact, the point spectrum of the Floquet operator does not ensure the boundedness of the function $E_\varphi^A(t)$ if, for example, $U(t,0)\psi$ escapes dom A for finite t. It seems that at present there is no comprehensive result concerning this question. We note that another definition of states of bounded energy has been proposed;[1,9,10] it is based on the spectral projections of A corresponding to large eigenvalues and avoids such domain questions: although some important relations between that definition and $\mathscr{H}_{\mathrm{po}}(U)$ and $\mathscr{H}_{\mathrm{cont}}(U)$ have been found,[1,9,10] here we restrict ourselves to the more natural sets $\mathscr{S}^{\mathrm{bd}}(A)$ and $\mathscr{S}^{\mathrm{un}}(A)$.

According to the above-quoted results and references, for periodic and autonomous quantum systems we have

$$\mathscr{H} = \mathscr{H}^{\mathrm{p}}(U) \oplus \mathscr{H}^{\mathrm{f}}(U) \tag{4}$$

This relation also holds in the quasiperiodic case,[9] with U standing for the generalized Floquet operator acting on an enlarged Hilbert space.

It seems we have a rather clear characterization of the stable $\mathscr{H}^{\mathrm{p}}(U)$ and unstable $\mathscr{H}^{\mathrm{f}}(U)$ subspaces for periodic and autonomous systems, and recent advances toward such a classification for the case of quasiperiodic potentials. What can be said about general time dependence? Certainly the situation is more involved, and the example presented in Section 3 indicates some difficulties one may face in such studies.

It is worth mentioning that for Hamiltonians quadratic in positions and momenta with time-periodic coefficients, the possible asymptotic behaviors—in the spectral and, consequently, also in the topological sense—have been completely catalogued.[24]

In Section 2 some basic properties of the sets that appeared in the above definition are summarized. In particular, it is shown that for general time dependence $\operatorname{dom} A \cap \mathscr{H}^{\mathrm{f}}(U) \subset \mathscr{S}^{\mathrm{un}}(A)$ and $\mathscr{H}^{\mathrm{p}}(U) \supset \mathscr{S}^{\mathrm{bd}}(A)$. In Section 3 it is shown, by considering a simple (nonphysical) example, that there are cases in which relation (4) does not hold; the autocorrelation measures of the system are computed and we have indications that the use of the spectral properties of possible generalizations of the quasienergy and/or Floquet operators will not be as direct as for autonomous, periodic and quasiperiodic quantum systems.

2. SOME BASIC PROPERTIES

To begin with, we underline that in this section $U(t, 0)$ denotes a propagator acting on $\mathscr{H}$, which we suppose can be nonperiodic.

Lemma 2. (i) If $\varphi \in \mathscr{H}^{\mathrm{f}}(U)$, then for any compact operator $\mathbf{K}: \mathscr{H} \to \mathscr{H}$,

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \|\mathbf{K} U(t, 0) \varphi\| \, dt = 0$$

(ii) $\mathscr{H}^{\mathrm{f}}(U)$ is a closed subspace of $\mathscr{H}$.

(iii) $\mathscr{H}^{\mathrm{p}}(U)$ is a closed subspace of $\mathscr{H}$.

Proof. (i) Pick $\varphi \in \mathscr{H}^{\mathrm{f}}(U)$ and $\mathbf{K}$ a compact operator. For each $\tau > 0$ fixed, we can apply the Schwarz inequality to get

$$\int_0^\tau \|\mathbf{K} U(t, 0) \varphi\| \, dt \leqslant \sqrt{\tau} \left[\int_0^\tau \|\mathbf{K} U(t, 0) \varphi\|^2 \, dt \right]^{1/2}$$

Then,

$$\frac{1}{\tau} \int_0^\tau \|\mathbf{K} U(t, 0) \varphi\| \, dt \leqslant \left[\frac{1}{\tau} \int_0^\tau \|\mathbf{K} U(t, 0) \varphi\|^2 \, dt \right]^{1/2}$$

and (i) follows.

(ii) It is trivial to see that $\mathscr{H}^{\mathrm{f}}(U)$ is a linear manifold. Let $\varphi \in \overline{\mathscr{H}^{\mathrm{f}}(U(t,0))}$ and $\mathbf{K}$ be a compact operator. Given $\varepsilon > 0$, pick $\psi \in \mathscr{H}^{\mathrm{f}}(U)$ such that $\|\psi - \varphi\| < \varepsilon$. Thus

$$\begin{aligned}
&\frac{1}{\tau}\int_0^\tau \|\mathbf{K}U(t,0)\varphi\|^2\,dt \\
&\quad = \frac{1}{\tau}\int_0^\tau \|\mathbf{K}U(t,0)(\varphi-\psi) + \mathbf{K}U(t,0)\psi\|^2\,dt \\
&\quad \leqslant \frac{1}{\tau}\int_0^\tau [\|\mathbf{K}U(t,0)(\varphi-\psi)\| + \|\mathbf{K}U(t,0)\psi\|]^2\,dt \\
&\quad \leqslant \frac{1}{\tau}\int_0^\tau [\|\mathbf{K}\|\cdot\|(\varphi-\psi)\| + \|\mathbf{K}U(t,0)\psi\|]^2\,dt \\
&\quad \leqslant \|\mathbf{K}\|^2\,\|\varphi-\psi\|^2 + \frac{1}{\tau}\int_0^\tau \|\mathbf{K}U(t,0)\psi\|^2\,dt + 2\,\|\mathbf{K}\|^2\,\|\psi\|\cdot\|(\varphi-\psi)\|
\end{aligned}$$

If τ is sufficiently large,

$$\frac{1}{\tau}\int_0^\tau \|\mathbf{K}U(t,0)\psi\|^2\,dt < \varepsilon$$

and one immediately sees that $\varphi \in \mathscr{H}^{\mathrm{f}}(U)$. This implies that $\mathscr{H}^{\mathrm{f}}(U)$ is closed; (ii) is proven.

(iii) The proof is outlined in the work of Enss and Veselic,[1] p. 164. ▮

Lemma 3.

$$\mathscr{H}^{\mathrm{p}}(U) \perp \mathscr{H}^{\mathrm{f}}(U)$$

Proof. Let $\varphi \in \mathscr{H}^{\mathrm{f}}(U)$ and $\pi \in \mathscr{H}^{\mathrm{p}}(U)$. Then

$$\langle \varphi \mid \pi \rangle = \frac{1}{\tau}\int_0^\tau \langle \varphi \mid \pi \rangle\,dt = \frac{1}{\tau}\int_0^\tau \langle U(t,0)\varphi \mid U(t,0)\pi \rangle\,dt$$

For each $\varepsilon > 0$ there exists P_ε that projects onto a finite-dimensional subspace of $\mathscr{H}$ such that, for all $t \geqslant 0$, $\|(I - P_\varepsilon)\,U(t,0)\pi\| < \varepsilon/(2\|\varphi\|)$; thus

$$\begin{aligned}
\langle \varphi \mid \pi \rangle &= \frac{1}{\tau}\int_0^\tau \langle (P_\varepsilon + I - P_\varepsilon)\,U(t,0)\varphi \mid U(t,0)\pi \rangle\,dt \\
&= \frac{1}{\tau}\int_0^\tau \langle P_\varepsilon U(t,0)\varphi \mid U(t,0)\pi \rangle\,dt \\
&\quad + \frac{1}{\tau}\int_0^\tau \langle (I - P_\varepsilon)\,U(t,0)\varphi \mid U(t,0)\pi \rangle\,dt
\end{aligned}$$

We see at once that

$$|\langle \varphi \mid \pi \rangle| \leqslant \|U(t,0)\pi\| \frac{1}{\tau}\int_0^\tau \|P_\varepsilon U(t,0)\varphi\|\, dt$$

$$+ \|U(t,0)\varphi\| \frac{1}{\tau}\int_0^\tau \|(I-P_\varepsilon)\, U(t,0)\pi\|\, dt$$

Since $\varphi \in \mathscr{H}^{\mathrm{f}}(U)$, by Lemma 1(i),

$$\frac{1}{\tau}\int_0^\tau \|P_\varepsilon U(t,0)\varphi\|\, dt < \varepsilon/(2\,\|\pi\|)$$

if τ is large enough. Therefore, $|\langle \varphi \mid \pi \rangle| < \varepsilon$ for any $\varepsilon > 0$, and Lemma 3 is proven. ∎

Proposition 4. Let A be as defined in the last section; then:

(i) $\mathscr{H}^{\mathrm{p}}(U) \supset \mathscr{S}^{\mathrm{bd}}(A)$

(ii) $\operatorname{dom} A \cap \mathscr{H}^{\mathrm{f}}(U) \subset \mathscr{S}^{\mathrm{un}}(A)$

Proof. (i) Let $\varphi \in \mathscr{S}^{\mathrm{bd}}(A)$; then there exists $M > 0$ such that $\langle U(t,0)\varphi \mid AU(t,0)\varphi \rangle \leqslant M$ for all $t \geqslant 0$. Since (φ_n) is a basis of $\mathscr{H}$, we can write $\varphi(t) \equiv U(t,0)\varphi = \sum_n a_n(t)\varphi_n$; then we get

$$\lambda_N \sum_{n \geqslant N} |a_n(t)^2| \leqslant \sum_n \lambda_n\, |a_n(t)^2| \leqslant M, \qquad \forall t \geqslant 0$$

For each $\varepsilon > 0$ there is $N(\varepsilon)$ such that

$$\sum_{n=N(\varepsilon)+1}^{\infty} |a_n(t)|^2 \leqslant \varepsilon\, \|\varphi\|^2, \qquad \forall t \geqslant 0$$

If $B_{N(\varepsilon)}$ denotes the projection operator onto the subspace spanned by $(\varphi_n)_{n=1}^{N(\varepsilon)}$, we have

$$\|B_{N(\varepsilon)}\varphi(t) - \varphi(t)\|^2 = \sum_{n=1+N(\varepsilon)}^{\infty} |a_n(t)|^2 < \varepsilon\, \|\varphi\|^2, \qquad \forall t \geqslant 0$$

Since ε is arbitrary and $B_{N(\varepsilon)}$ projects onto a finite-dimensional subspace, we have $\varphi \in \mathscr{H}^{\mathrm{p}}(U)$.

(ii) The proof is immediate from (i) above, Lemma 3, and $\operatorname{dom} A = \mathscr{S}^{\mathrm{un}}(A) \oplus \mathscr{S}^{\mathrm{bd}}(A)$; indeed,

$$\mathscr{S}^{\mathrm{un}}(A) \supset \operatorname{dom} A \cap [\mathscr{H}^{\mathrm{p}}(U)]^{\perp} \supset \operatorname{dom} A \cap \mathscr{H}^{\mathrm{f}}(U) \quad ∎$$

3. THE U-UNUSUAL SUBSPACE

In this section we present an example for which relation (4) does not hold. Although rather peculiar, this example illustrates the possible "unusual" properties (see Section 4) of nonstationary quantum systems. We then define the "U-unusual" subspace $\mathscr{H}^{a}(U)$ by the relation

$$\mathscr{H} = \mathscr{H}^{p}(U) \oplus \mathscr{H}^{f}(U) \oplus \mathscr{H}^{a}(U)$$

Example. Let (ε_n), $n \geqslant 1$, be a sequence with $\varepsilon_n \in \{-1, 0, 1\}$. The model is given by the kicked Hamiltonian

$$H = p^2 + x \sum_{n=1}^{\infty} \varepsilon_n \delta(t-n), \qquad x \in [0, 2\pi) \tag{5}$$

acting on $\mathscr{H} = L^2(\mathbb{T})$. The temporal evolution operator until $t = (n-0)$, $n \in \mathbb{N}$ [just before the $(n+1)$th kick], is given by

$$U(n, 0) \equiv V_n V_{n-1} \cdots V_1$$

where $V_j = e^{-ip^2} e^{-i\varepsilon_j x}$. Between two consecutive kicks we have free time evolution. This model is constituted of jumps (or no jump at all) between eigenstates of the free evolution operator, and our goal is to pick the consecutive jumps properly. If we denote the eigenstate of the unperturbed Hamiltonian p^2 by $\psi_s = e^{-isx}$ and $\Gamma_j = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_j$, we have

$$U(n, 0)\psi_s = \exp\left[-i \sum_{j=1}^{n} (\Gamma_j + s)^2\right] \exp[-i(\Gamma_n + s)x]$$

$$\langle U(n, 0)\psi_s \mid p^2 U(n, 0)\psi_s \rangle = (\Gamma_n + s)^2$$

Set $u_n = 4(10^n + 10^{n-1} + \cdots + 10)$ for $n \geqslant 1$, $u_0 = 0$, and choose

$$\varepsilon_j \equiv \begin{cases} 1, & j \in A_n \equiv [u_n + 1, u_n + 10^{n+1}) \\ -1, & j \in B_n \equiv [u_n + 1 + 10^{n+1}, u_n + 2 \times 10^{n+1}) \\ 0, & j \in C_n \equiv [u_n + 1 + 2 \times 10^{n+1}, u_{n+1}) \end{cases} \tag{6}$$

Since $\varepsilon_j = 0$ iff $j \in C_n$ and ($\# Z$ denotes the cardinality of the set Z)

$$\frac{\# C_n}{\# A_n + \# B_n + \# C_n} = \frac{1}{2}$$

it follows that

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \|P_s U(t, 0)\psi_s\|^2 \, dt = \frac{1}{2}$$

where P_s is the (compact) projection operator onto the subspace spanned by ψ_s. Therefore, $\psi_s \notin \mathscr{H}^{\mathrm{f}}(U)$, and since $\{\psi_s : s \in \mathbb{Z}\}$ is a basis of $\mathscr{H}$ and by Lemma 2(iii) $\mathscr{H}^{\mathrm{f}}(U)$ is closed, it follows that $\mathscr{H}^{\mathrm{f}}(U) = \{0\}$.

Since for each $m \in \mathbb{N}$, $m \geqslant s$, there exists $j \in \mathbb{N}$ such that $U(j, 0)\psi_s = e^{i\theta(s,m)}\psi_m$ for some phase factor $\theta(s, m)$, the set $\{U(t, 0)\psi_s : t \geqslant 0\}$ is not precompact; then $\psi_s \notin \mathscr{H}^{\mathrm{p}}(U)$ and one can conclude that $\mathscr{H}^{\mathrm{p}}(U) = \{0\}$. Therefore $\mathscr{H} = \mathscr{H}^{\mathrm{a}}(U)$.

In order to clarify the nature of this "unusual" behavior we compute the autocorrelation measures[3,6,25] for the eigenvectors ψ_s. For kicked systems we have

$$C_\psi(m) = \lim_{N \to \infty} \frac{1}{N} \sum_{p=0}^{N-1} \langle U(p, 0)\psi \mid U(p+m, 0)\psi \rangle$$

and the autocorrelation measure σ_ψ is given by relation (1).

A direct calculation leads to

$$C_{\psi_s}(m) = \lim_{N \to \infty} \frac{1}{N} \sum_{p=1}^{N} \delta_{\Gamma_p, \Gamma_{p+m}} \exp\left[-i \sum_{j=1+p}^{p+m} (\Gamma_j + s)^2\right]$$

We have $\Gamma_p = \Gamma_{p+m}$ only in two situations. The first one is when both p and $(p+m)$ belong to C_n for some $n(\Gamma_p = \Gamma_{p+m} = 0)$, and for large N this occurs approximately $N/2$ times. The second situation occurs only for even m and when $p = (u_n - m/2)$, which happens once for each C_n. Since $n/u_n \to 0$ as $n \to \infty$, this term does not contribute to $C_{\psi_s}(m)$.

Taking into account that $\varepsilon_j = 0$ if $j \in C_n$, we obtain

$$\begin{aligned} C_{\psi_s}(m) &\cong \frac{1}{u_N} \sum_{n=1}^{N} \sum_{\left(\substack{p \in C_n \\ (p+m) \in C_n}\right)} \delta_{\Gamma_p, \Gamma_{p+m}} \exp\left(-i \sum_{j=1+p}^{p+m} s^2\right) \\ &= \frac{1}{u_N} \sum_{n=1}^{N} (\# C_n - m) \exp(-ims^2) \\ &= [\exp(-ims^2)] \left[\sum_{n=1}^{N} (\# C_n) - Nm\right] \Big/ u_N \xrightarrow{N \to \infty} [e(-ims^2)]/2 \end{aligned}$$

Then we get

$$\sigma_{\psi_s} = \tfrac{1}{2}(\ell + \delta_{s^2})$$

where ℓ denotes the Lebesgue measure and δ_{s^2} is the unit mass at s^2 (mod 2π). From this we see that the autocorrelation measure σ_{ψ_s} is the sum of a point part and a continuous part for all ψ_s. This gives us some insight about $\mathscr{H}^{\mathrm{a}}(U)$. Notice that for any $\varphi \in \operatorname{dom} p^2$, $E_\varphi^{p^2}(t)$ is an unbounded function of time t.

Summing up

Proposition 5. For the model (5)–(6) we have $\mathscr{H} = \mathscr{H}^{a}(U) = \mathscr{S}^{un}(p^2)$, and for any ψ_s its autocorrelation measure is the sum of a pure point and a pure absolutely continuous part.

Remarks. (i) The above example shows that $[\mathrm{dom}\, A \cap \mathscr{H}^{f}(U)] \neq \mathscr{S}^{un}(A)$ in some cases [see Proposition 4(ii)]. At the moment I cannot say if the equality in Proposition 4(i) holds, although there are examples showing that the equality does not hold if $U(t, 0)$ is a family of nonunitary isometries.

(ii) We have found that for ψ_s the autocorrelation measure is not pure, i.e., it is neither pure continuous nor pure point. Therefore we have a strong indication that the spectral properties of generalized Floquet and quasienergy operators are of little interest for the stability problem of general nonautonomous systems, since for unitary and self-adjoint operators acting in a Hilbert space $\mathscr{K}$ the spectral measures are pure point for vectors in $\mathscr{K}_{po}$ and pure continuous for vectors in $\mathscr{K}_{cont}$; moreover, $\mathscr{K} = \mathscr{K}_{po} \oplus \mathscr{K}_{cont}$ (see Section 4).

4. CONCLUSIONS

The problem of characterizing the stable and unstable subspaces of driven systems is not yet well understood. We have reported briefly four possible approaches to this problem that have appeared in the literature, and analyzed more closely the generalization of the notion of point spectral subspace to systems governed by time-dependent Hamiltonians advocated by Enns and Veselic,[1,10] i.e., by using the notion of precompact trajectories.

Another possible approach to the problem is to consider the time behavior of autocorrelation functions and the corresponding autocorrelation measures. In the cases of autonomous and time-periodically driven systems the autocorrelation measures coincide with the spectral measures of the corresponding Hamiltonian or Floquet operator. In the case of systems driven by some classical dynamic systems with an ergodic invariant measure (which includes the quasiperiodic case[9]), some (expected) relations between the spectrum of the generalized quasienergy operator and the asymptotic behavior of autocorrelation functions have been found.[8,9]

Although it has been difficult to find analytical solutions and even to perform reliable numerical computations of the autocorrelation measures,[5-7,25] the ad hoc example of Section 3, with $\mathscr{H} = \mathscr{H}^{a}(U)$, was amenable to analytical treatment. Certainly, the "mixed" autocorrelation measures are not new; for instance, Combescure[5,6] has computed the autocorrelation measures for both two-level systems and harmonic oscillators subjected to aperiodic kicks modulated according to the Thue–Morse sequence,[25] and found that for some parameter values the autocorrelation measure is the sum of a pure point part and a pure singular continuous part for any nontrivial vector in the Hilbert space. However, there are indications that Combescure's results hold only for parameters in a set of zero Lebesgue measure,[26] and the relation to the topological characterization of Enss and Veselic is not clear. The fact that the autocorrelation measure is the sum of a continuous part and a point part for any vector in the Hilbert space cannot happen for autonomous or periodically driven systems, and this also makes the use of the spectral properties of some kind of quasienergy operator for the study of the quantum stability of general driven systems questionable.

Finally, we would like to mention that perhaps the adjective "unusual" used to classify $\mathscr{H}^{a}$ may become inappropriate; it might, for instance, be replaced with "usual" in case someone is able to prove that $\mathscr{H} = \mathscr{H}^{a}$ is the rule for driven quantum systems.

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